# Highly Nonlinear Boolean Functions with Optimal Algebraic Immunity

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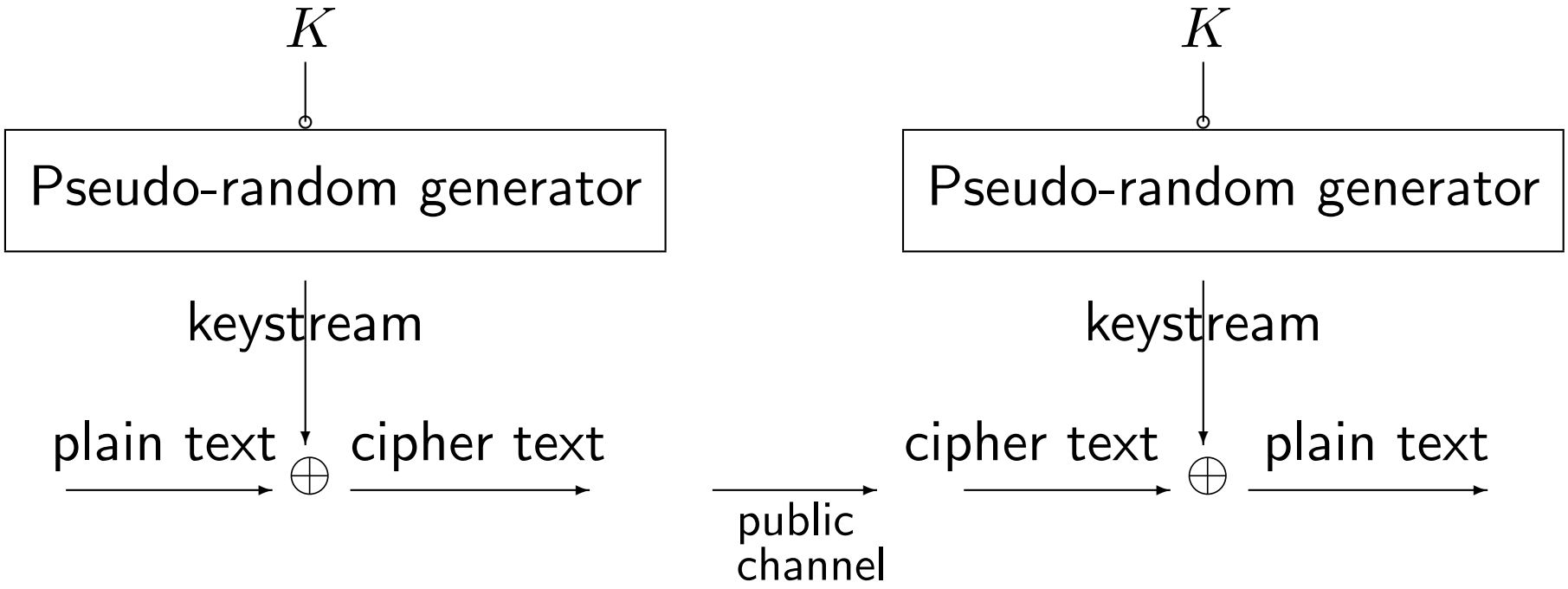
# Outline

- Preliminaries on stream ciphers and Boolean functions
- Algebraic attacks on stream ciphers and algebraic immunity
- The known Boolean functions with optimal algebraic immunity
- Recent developments

# Preliminaries on stream ciphers and Boolean functions

**Synchronous stream ciphers** :



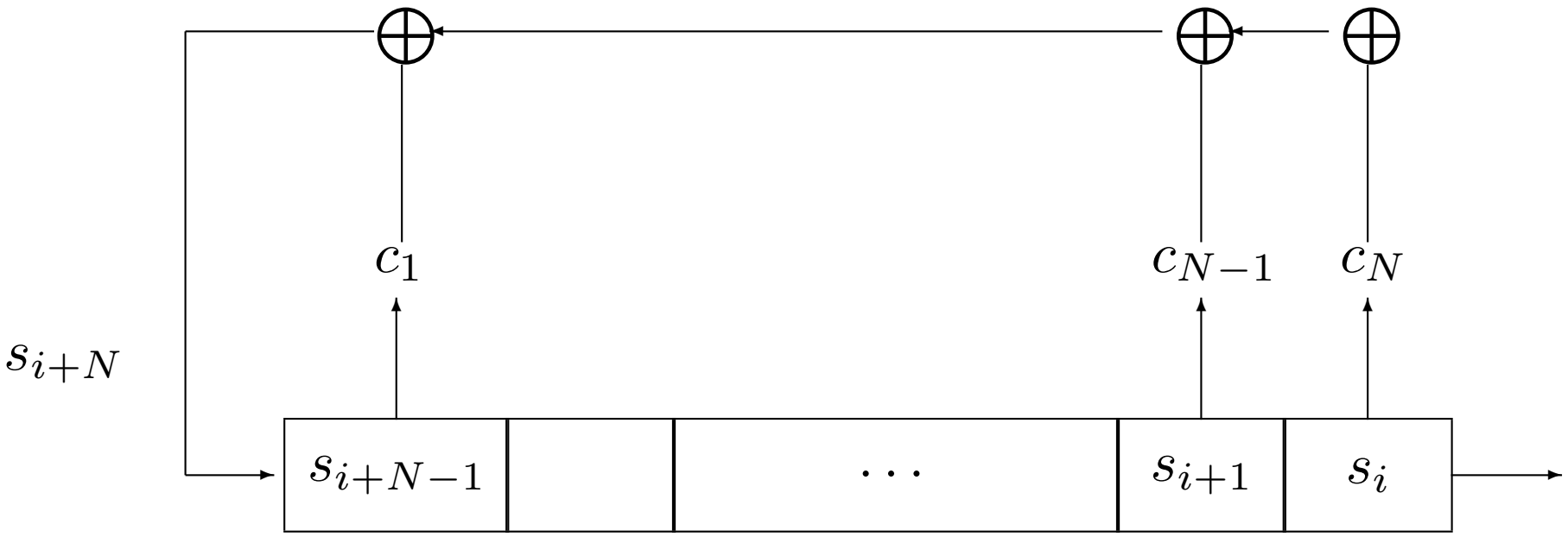
Every PRG consists in a linear part (for efficiency) and a nonlinear part (for robustness).

**Boolean functions** $f : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ are often used in the nonlinear part.

There exist **two theoretical models** for their use in the pseudo-random generators (PRG) of Synchronous stream ciphers.

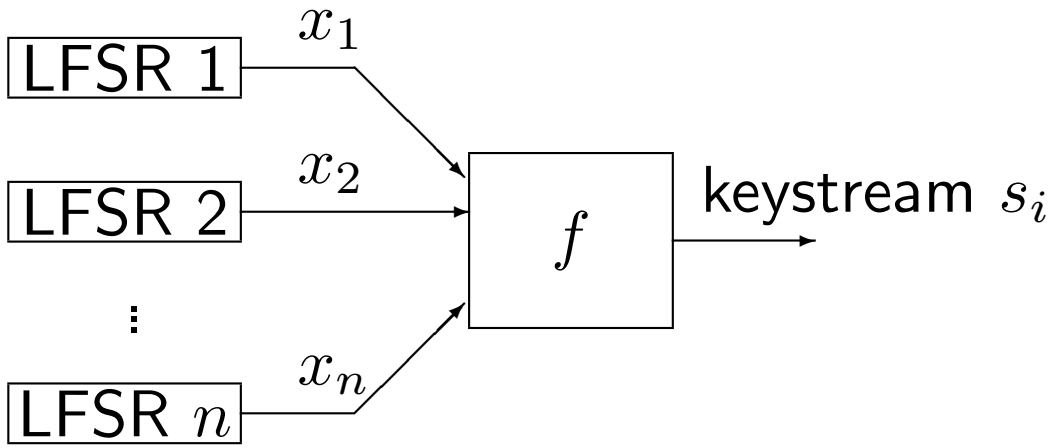
Both use Linear Feedback Shift Registers in the linear part :

*Linear feedback shift registers* :

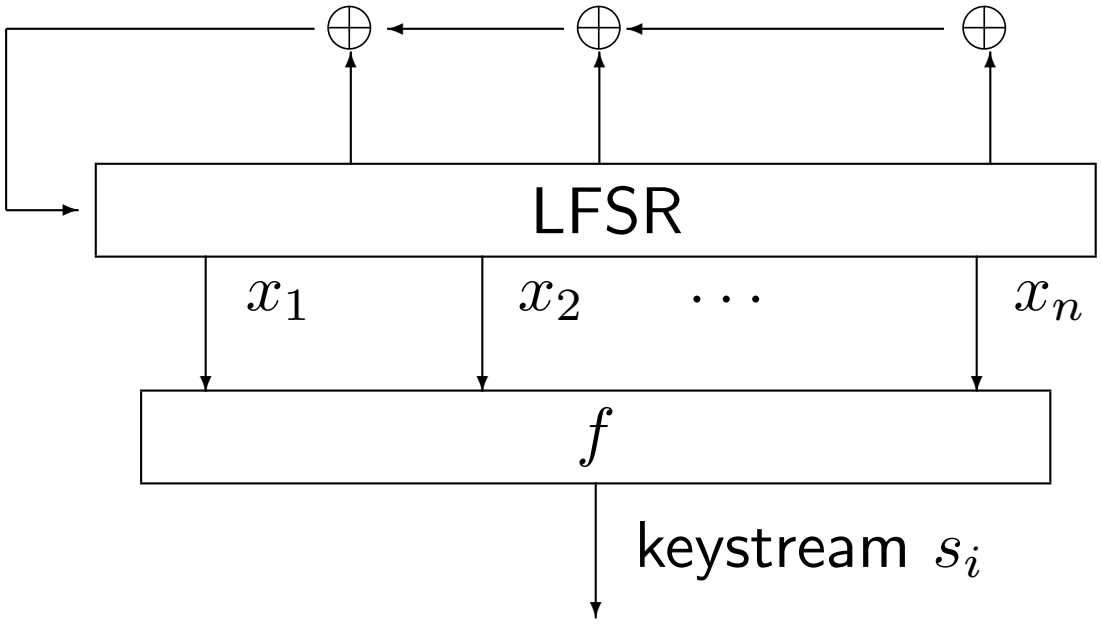



$s_i = \sum_{j=1}^{N} c_j s_{i-j}$.

*Combiner model* :



*Filter model*



In both models, $f$ must be balanced to avoid distinguishing attacks.

**Two representations of Boolean functions** :

- *The Algebraic Normal Form* (ANF) :

$$f(x_1, \cdots, x_n) = \sum_{I \subseteq \{1,\ldots,n\}} a_I \left( \prod_{i \in I} x_i \right), \; a_I \in \mathbb{F}_2.$$

The ANF exists and is unique.

*The algebraic degree* is the degree of the ANF.
It must be large because of Berlekamp-Massey and Rønjom-Helleseth attacks.

*Affine* functions : sums of linear functions and constants :
$a_1\, x_1 + \cdots + a_n\, x_n + \epsilon = a \cdot x + \epsilon\ ; \quad a \in \mathbb{F}_2^n; \quad \deg \leq 1.$
Their set is the Reed-Muller code of order 1.

- *The univariate representation (the trace representation)* :

- The vector space $\mathbb{F}_2^n$ is endowed with the structure of the field $\mathbb{F}_{2^n}$.
Any function $f : \mathbb{F}_{2^n} \mapsto \mathbb{F}_{2^n}$ admits the unique representation :

$$f(x) = \sum_{j=0}^{2^n-1} a_j\, x^j; \quad a_j,\, x \in \mathbb{F}_{2^n}.$$

- $f$ is Boolean if and only if :

$$a_0, a_{2^n-1} \in \mathbb{F}_2 \text{ and } a_{2j} = (a_j)^2, \forall j \in \mathbb{Z}/(2^n-1)\mathbb{Z}.$$

Hence :

$$f(x) = tr(P(x)), \text{ where } tr(x) = x + x^2 + x^{2^2} + \cdots + x^{2^{n-1}}.$$

Then the algebraic degree equals : $\max\{w_2(j);\ j \text{ s.t. } a_j \neq 0\}$, where $w_2(j)$ is the Hamming weight of the binary expansion of $j$.

Affine functions $tr(ax) + \epsilon$, $a \in \mathbb{F}_2^n$, $\epsilon \in \mathbb{F}_2$.

The *Walsh transform* of a Boolean function :

$$\widehat{\mathrm{f}}(a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+a\cdot x} \text{ or } \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+tr(ax)}.$$

The *Hamming distance* between two functions :

$$d_H(f,g) = w_H(f+g) = |\{x \in \mathbb{F}_2^n \,/\, f(x) \neq g(x)\}.$$

The *nonlinearity* of a Boolean function $f$ is the minimum Hamming distance from $f$ to affine functions (i.e. its distance to the Reed-Muller code of order 1) and equals :

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |\widehat{\mathrm{f}}(a)|.$$

The nonlinearity $nl$ is upper bounded by $2^{n-1} - 2^{n/2-1}$ (covering radius bound). This maximum is achieved by bent functions.

The nonlinearity $nl$ must be large to prevent the system from fast correlation attacks.

Balancedness, high algebraic degree and large nonlinearity was considered as roughly sufficient for the filter model of pseudo-random generator before the introduction of algebraic attacks.

# Algebraic attacks on stream ciphers and algebraic immunity

**Algebraic attacks** : *Principle* (Shannon) :

-Find equations with the key bits as unknowns
-Solve the system of these equations.

For stream ciphers (combiner model and filter model) :
- denote by $(s_0, \ldots, s_{N-1})$ the initial state of the linear part of the pseudo-random generator ;
- there exists a linear automorphism $L$ and a linear mapping $L'$ s.t.

$$s_i = f(L' \circ L^i(s_0, \ldots, s_{N-1})).$$

*Problem of the general algebraic attack :*

Highly non-linear equations with many unknowns.

*But with stream ciphers* we can have many equations $\longrightarrow$

over-defined system.

One can then linearize the system (or use Gröbner bases).

*However the number of unknowns is then much too large*.

Courtois-Meier : If one can find $g \neq 0$ and $h$ of low degrees such that $fg = h$, then the equation $s_i = f(L' \circ L^i(s_0, \ldots, s_{N-1}))$ implies the low degree equation :

$$s_i \; g(L' \circ L^i(s_0, \ldots, s_{N-1})) = h(L' \circ L^i(s_0, \ldots, s_{N-1}))$$

and the degree of the nonlinear system and the number of unknowns in the related linear system decrease.

**Algebraic immunity** :

A necessary and sufficient condition for existence of low degree $g \neq 0$ and $h$ such that $fg = h$ (Meier-Pasalic-C.C.) :
there exists $g \neq 0$ of low degree such that $fg = 0$ or $(f + 1)g = 0$.

*Definition* : a function $g$ such that $fg = 0$ is called an $annihilator$. The $algebraic\ immunity$ $AI(f)$ is the minimum degree of the nonzero annihilators of $f$ and of those of $f + 1$.

Related to coding problems over the erasure channel.

We have : $AI(f) \leq \deg(f)$ and $AI(f) \leq \left\lceil \frac{n}{2} \right\rceil$.

A variant of algebraic attacks, called "fast algebraic attack" needs the existence of $g \neq 0$ and $h$ such that $fg = h$, where only $g$ has low degree and $h$ has degree significantly lower than $n$.

# The known Boolean functions with optimal algebraic immunity

**$\leq$ 2008 :**

- The majority function defined by :

$$f(x) = 1 \text{ iff } w_H(x) \geq n/2.$$

and its generalizations by Dalai et al., Bracken, C.C... ;

- An iterative construction (Dalai-Gupta-Maitra), $n$ even.

These functions have high degree but *insufficient nonlinearity* and bad resistance to Fast Algebraic Attacks (Dalai, Gupta, Maitra, Armknecht, C.C., Gaborit, Meier, Ruatta...).

**2008 :**

**Definition** [CF function]

*Let $n \geq 2$ and $\alpha$ a primitive element of the field $\mathbb{F}_{2^n}$.*
*We denote by $f$ the Boolean function on $\mathbb{F}_{2^n}$ whose support is $\{\alpha^s, \cdots, \alpha^{2^{n-1}+s-1}\}$.*

**Theorem** (Feng, Liao, Yang)

*The function $f$ defined above has optimal algebraic immunity $\lceil n/2 \rceil$.*

A simpler proof by C.C., Feng uses the BCH bound (on cyclic codes).

**Algebraic degree** (C.C., Feng) : $f$ has degree $n-1$ (optimal).

**Nonlinearity** (Brandstätter, Lange, Winterhof, Hakala, Nyberg, C.C., Feng, Tang, C.C.) :

$$nl(f) \geq 2^{n-1} - \left(\frac{n \ln 2}{\pi} + 0.74\right) 2^{\frac{n}{2}} - 1$$

(better but still not sufficient).

The values of $nl(f)$ computed for $n \leq 24$ are good.

The **immunity to fast algebraic attacks** checked for $n \leq 10$ is good.

A forthcoming paper by M. Liu, Y. Zhang, and D. Lin proves it.

The function can be fastly evaluated by the Pohlig-Hellman method : one output bit per cycle with 40,000 transistors for $n = 20$.

# Recent developments

**Definition** (Z. Tu and Y. Deng - Designs, Codes and Cryptography)

$$(x,y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n};\ f^{\#}(x,y) = f(xy^{2^n-2}) = f\left(\frac{x}{y}\right),\ \text{with } \frac{x}{0} = 0.$$

**Theorem** (Z. Tu and Y. Deng) *up to a conjecture* (studied by J.-P. Flori, H. Randriambololona, G. Cohen 1 and S. Mesnager)

*The function $f^{\#}$ has optimal algebraic immunity $n$.*

**Nonlinearity** :

$$nl(f^{\#}) = 2^{2n-1} - 2^{n-1}$$

($f^{\#}$ has best possible nonlinearity ; it is bent).

**Remark**. Function $f^{\#}$ is not balanced and has degree at most $n$ (as any bent function). But the function :

$$f^{\#'}(x,y) = \begin{cases} f\left(\frac{x}{y}\right) \text{ if } y \neq 0 \\ f(x) \text{ if } y = 0 \end{cases}$$

has optimal algebraic immunity as well and is balanced. Its degree equals $2n-1$ and $nl(f^{\#'}) \geq 2^{2n-1} - 2^{n-1} - n\,2^{n/2}\ln 2 - 1$.

**But observations** :

- This function is weak against the fast algebraic attack (C.C., IACR ePrint Archive).

- Its distance to bent functions and therefore to functions of algebraic degrees at most $n/2$ is small and this implies that its resistance to fast algebraic attack is weak (Wang-Johansson, INSCRYPT 2010).

Any function constructed with a similar method would have the same drawback.

**Definition** (D. Tang, C.C., X. Tang)

$$n \geq 2;\ (x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n}; \qquad f_1(x, y) = f(xy).$$

**Algebraic immunity** : *up to a conjecture* : $AI(f_1) = n$.
This conjecture has later been proved by Cohen and Flori, using ideas common with H. Randriambololona and S. Mesnager.

**Algebraic degree** : $2n - 2$.

The **immunity to fast algebraic attacks** checked for $n \leq 8$ is good.

**Nonlinearity** : $N_{f_1} > 2^{2n-1} - \left(\frac{\ln 2}{\pi} n + 0.42\right) 2^n - 1$.

**Slight modification to get balanced functions** :

Let $n = 2^t m$ be an even integer no less than $4$ such that $t \geq 1$ and $\gcd(m, 2) = 1$.

$$f_2(x, y) = \begin{cases} f_1(x, y), & x \neq 0 \\ u(y), & x = 0 \end{cases}$$

where $u$ is balanced on $\mathbb{F}_{2^n}$ satisfying $u(0) = 0$, $\deg(u) = n - 1$, and $\max_{a \in \mathbb{F}_{2^n}} |W_u(a)| \leq 2^{\frac{m+1}{2}}$ if $t = 1$ and $\max_{a \in \mathbb{F}_{2^k}} |W_u(a)| \leq \sum_{i=1}^{t-1} 2^{\frac{n}{2^{i+1}}} + 2^{\frac{m+1}{2}}$ if $t \geq 2$ ($u$ does exist).

*A further modification (more complex) allows acheiving 1-resiliency.*

**Algebraic degree and algebraic immunity** $f_2$ has maximal algebraic degree for balanced function and optimal algebraic immunity.

**Immunity to fast algebraic attacks, Nonlinearity** : similar to $f_1$.

*The Tu-Deng conjecture has been further generalized by C.C.-Tang-Tang and a related construction has been proposed by Jin, Liu and Wu.*

# The exact values of the nonlinearity

| $n$ | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|
| $2^{n-1}-2^{n/2}$ | 4 | 24 | 112 | 480 | 1984 | 8064 |
| $\mathcal{N}_{CF}$ | 4 | 24 | 112 | 484 | 1970 | 8036 |
| $\mathcal{N}_{f_2}$ | 4 | 22 | 108 | 476 | 1982 | 8028 |

| $n$ | 16 | 18 | 20 | 22 | 24 | 26 |
|---|---|---|---|---|---|---|
| $2^{n-1}-2^{n/2}$ | 32512 | 130560 | 523264 | 2095104 | 8384512 | 33546240 |
| $\mathcal{N}_{CF}$ | 32530 | 130442 | 523154 | 2094972 | 8384536 | 33545716 |
| $\mathcal{N}_{f_2}$ | 32508 | 130504 | 523144 | 2094980 | 8384490 | 33545992 |

| $n$ | 28 | 30 | 32 | 34 | 36 | 38 |
|---|---|---|---|---|---|---|
| $2^{n-1}-2^{n/2}$ | 134201344 | 536838144 | 2147418112 | 8589803520 | 34359476224 | 137438429184 |
| $\mathcal{N}_{f_2}$ | 134201460 | 536838052 | 2147416552 | 8589818968 | 34359469052 | 137438441620 |